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System of N Thin Coaxial Lenses

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***Abstract:** In geometrical optics, in a system of two thin coaxial lenses, there are several standard formulas, including "$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$". The purpose of this paper is to generalize these formulas to the case of a system of an arbitrary number of thin lenses. In particular, this paper proves that the focal length F_n of a system of n thin coaxial lenses is given by $\frac{1}{F_n} = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n; d_n=1}}^{a_s} \frac{1}{f_{r_s}}\right) d_{a_s}\right]\right\}$, where, f_r is the focal length of the r^{th} lens, and d_r is the distance between the r^{th} lens and $(r+1)^{th}$ lens. For a fixed value of m, all combinations of values of the a's (satisfying the condition "$0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = n$") are taken in the inner sum.*

***Keywords:** coaxial lens system, focal length, Gaussian lens equation, magnification formula*

I. INTRODUCTION

In this article, the term "lens(es)" is taken to mean "thin lens(es)". The diagram below shows a system of n lenses, in which the r^{th} lens is denoted by L_r. A ray of light AB, parallel to the principal (or optical) axis XY of the system, is refracted at B by the first lens, L_1, and emerges along BC. The ray is then refracted by subsequent lenses (of which only L_n is shown) and finally emerges from the system along EF to intersect XY at F. AB and EF intersect at D and DH is perpendicular to XY.

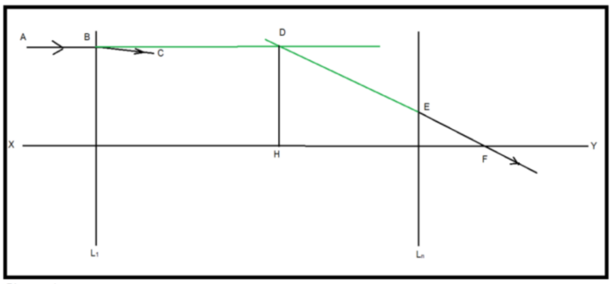

Diagram 1

The following are some (established) definitions.

- F is called the **image focus** (or **rear focal point** or **back focal point**) of the system.
- H is called the **Image Principal Point** (or **image unit point** or **second principal point** or **second unit point**) of the system.
- The distance between HF is called the **image focal length** of the system.

If the ray of light AB were to travel in the opposite direction (i.e. from right to left) and intersect L_n first and finally emerge from L_1, then there would be a corresponding

- object focus (or front focal point) of the system
- object principal point (or object unit point or first principal point or first unit point) of the system
- object focal length of the system

The reciprocal of the object focal length is called the object power of the system. Similarly, the reciprocal of the image focal length is called the image power of the system.

For a single lens and a system of two lenses, it has been established that the value of the object focal length and the value of the image focal length are the same.

II. SYSTEM OF 2 LENSES

The following notations are used in the formulas below for a system of two lenses.

- The power of
  - the first lens is k_1
  - the second lens is k_2
  - the lens system is K
- The focal length of the lens system is F
- The distance between
  - the lenses is d
  - the first principal point and the first lens is h_1
  - the second principal point and the second lens is h_2
  - an object and the first lens is u
  - the corresponding image and the second lens is v
- The transverse (or linear) magnification is m

The following are well-known formulas for a system of two lenses [1].

- $K = k_1 + k_2 - dk_1k_2$
- $h_1 = \frac{dk_2}{k_1 + k_2 - dk_1k_2}$
- $h_2 = \frac{dk_1}{k_1 + k_2 - dk_1k_2}$
- $\frac{1}{u + h_1} + \frac{1}{v + h_2} = \frac{1}{F}$
- $m = \frac{v + h_2}{F} - 1$

From the above formulas, if a system of two lenses

- of power k_1 and k_2
- and separated by a distance d apart

is replaced with a single lens

- of power $k_1 + k_2 - dk_1k_2$
- and positioned between the two lenses at a distance of h_2 from the second lens

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then, the usual formula "$\frac{1}{U}+\frac{1}{V}=\frac{1}{F}$" holds, where

- F is the focal length of the replacement lens
- V is the distance between the image and the replacement lens
- and U is the distance between the object and the first principal point.

This paper generalizes the above formulas to the case of a system of n lenses. Also, other results are established.

III. GENERALIZED FORMULA FOR THE OBJECT POWER

A. Notation Used

The following notations are used for a system of n lenses.

- The r^{th} lens is L_r
- The distance between L_r and L_{r+1} is d_r
- The focal length of the r^{th} lens is f_r
- The power of the r^{th} lens is k_r
- The (object) power of the lens system is K_n

B. Small Values of n

The formula for K_2 is derived using the equation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ together with a real image (considered a virtual object) and similar triangles. The formula is given by $K_2 = k_1 + k_2 - k_1d_1k_2 = [k_1 + k_2] - [(k_1)d_1(k_2)]$

By using this very method (i.e. virtual object and similar triangles) with three lenses (L_1, L_2, and L_3) the formula for K_3 can be obtained. Alternatively, the formula for K_3 can be obtained as follows.

L_1 and L_2 are replaced with a single equivalent lens, say L_{12}, so we are now dealing with 2 lenses, L_{12} and L_3. As mentioned above, L_{12} is of power $K_2 = k_1 + k_2 - d_1k_1k_2$ and is positioned between L_1 and L_2 at a distance of $\frac{d_1k_1}{k_1+k_2-d_1k_1k_2}$ from L_2. Thus, the distance D between L_{12} and L_3 is given by $D = d_2 + \frac{d_1k_1}{k_1+k_2-d_1k_1k_2}$.

Hence: $K_3 = K_2 + k_3 - DK_2k_3$

$= K_2 + k_3 - \left(d_2 + \frac{d_1k_1}{k_1+k_2-d_1k_1k_2}\right)K_2k_3$

$= (k_1 + k_2 - d_1k_1k_2) + k_3 - \left(d_2 + \frac{d_1k_1}{k_1+k_2-d_1k_1k_2}\right)(k_1 + k_2 - d_1k_1k_2)k_3$

$= (k_1 + k_2 - d_1k_1k_2) + k_3 - (d_2\{k_1 + k_2 - d_1k_1k_2\} + d_1k_1)k_3$

$= k_1 + k_2 + k_3 - d_1k_1k_2 - d_2(k_1 + k_2 - d_1k_1k_2)k_3 - d_1k_1k_3$

$= k_1 + k_2 + k_3 - d_1k_1k_2 - d_2(k_1 + k_2)k_3 + d_2d_1k_1k_2k_3 - d_1k_1k_3$

$= k_1 + k_2 + k_3 - d_1k_1k_2 - d_1k_1k_3 - d_2(k_1 + k_2)k_3 + d_2d_1k_1k_2k_3$

$$= [k_1 + k_2 + k_3] - [(k_1)d_1(k_2 + k_3) + (k_1 + k_2)d_2(k_3)] + [(k_1)d_1(k_2)d_2(k_3)]$$

Similarly, with four lenses the following formula for K_4 is obtained:

$$\begin{aligned} K_4 = {} & [k_1 + k_2 + k_3 + k_4] \\ & - [(k_1)d_1(k_2 + k_3 + k_4) + (k_1 + k_2)d_2(k_3 + k_4) + (k_1 + k_2 + k_3)d_3(k_4)] \\ & + [(k_1)d_1(k_2)d_2(k_3 + k_4) + (k_1)d_1(k_2 + k_3)d_3(k_4) + (k_1 + k_2)d_2(k_3)d_3(k_4)] \\ & - [(k_1)d_1(k_2)d_2(k_3)d_3(k_4)] \end{aligned}$$

C. Proposed Formula for K_n

In the formulas for K_2, K_3, and K_4 the terms having the same number of factors of the d's are grouped using square brackets. The sum of the terms in the $(m+1)^{th}$ pair of square brackets is denoted by T_m.

The following pattern seems to be developing:

- K_n is composed of T_0, T_1, …, T_{n-1}.
- The sign preceding T_m is $(-1)^m$.
- T_m is a sum with each summand being a product of the following factors: m d's and m+1 sums of k's, with each sum of k's enclosed in a pair of parentheses, ().

  Note the following:
- Some of these "sums of k's" may have only a single term.
- In each pair of parentheses, the index of the last k (except when not equal to n) equals that of the d which follows immediately.
- Each summand has m d factors out of a possible of n-1 d's. Thus, the number of summands in T_m is ${}^{n-1}C_m$ (a binomial coefficient).

A typical summand in T_m is $(k_1 + \cdots + k_{a_1})d_{a_1}(k_{a_1+1} + \cdots + k_{a_2})d_{a_2} \ldots (k_{a_{m-1}+1} + \cdots + k_{a_m})d_{a_m}(k_{a_m+1} + \cdots + k_n)$, where the a_r's are integers satisfying $1 \le a_1 < \ldots < a_m \le n-1$.

This typical summand

$= \{\prod_{s=1}^{m}(k_{a_{s-1}+1} + \cdots + k_{a_s})d_{a_s}\}\{k_{a_m+1} + \cdots + k_n\}$, where additionally $a_0 = 0$

$= \{\prod_{s=1}^{m}[(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s})d_{a_s}]\}\{\sum_{r_{m+1}=a_m+1}^{n} k_{r_{m+1}}\}$

$= \prod_{s=1}^{m+1}[(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s})d_{a_s}]$, where additionally $a_{m+1} = n$ and $d_{a_{m+1}} = 1$ (i.e. $d_n = 1$)

For a fixed m, by giving the a's all possible combinations of values satisfying $0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = n$, all the summands in T_m are obtained.

$$\text{Thus: } T_m = \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right)d_{a_s}\right]$$

$$\text{Hence: } K_n = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right)d_{a_s}\right]\right\}$$

D. Strategy for the Proof of the Proposed Formula for K_n

This formula will be proved to be the (object) power of the system by the Method of Mathematical Induction and by employing a strategy explained in this section.

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L_n will be replaced by two lenses $L`_n$ and $L_{n+1}$ (with power $k`_n$ and k_{n+1}, respectively) that are

- separated by a distance of d_n apart, so that the focal length of the system consisting of $L`_n$ and $L_{n+1}$ is equal to the focal length of $L_n$ (meaning that the system consisting of $L`_n$ and L_{n+1} is equivalent to L_n)
  That is: $k_n = k`_n + k_{n+1} - k`_n d_n k_{n+1}$
- appropriately positioned so that the object focal length of the new system of n+1 lenses is equal to the object focal length of the original system of n lenses (meaning that the system consisting of the n+1 lenses is equivalent to the system consisting of the n lenses)

L_{n-1} was at a distance of d_{n-1} from and to the left of L_n. The distance $d`_{n-1}$ (between $L_{n-1}$ and $L`_n$) will now be determined.

The diagram below shows a ray of light AC, parallel to the principal axis XY of L_n, being refracted at C and then intersects XY at F, the object focal point of L_n.

L_n is replaced with $L`_n$ and $L_{n+1}$ to maintain the same object focal point (F) and the same object focal length (HF). Therefore, the ray AB is now refracted at B by $L_{n+1}$ and is subsequently refracted at D by $L`_n$ to pass through F.

L_{n+1}, L_n and $L`_n$ intersect XY at P, H, and Q, respectively. BD intersects XY at G and DF intersects AB at C.

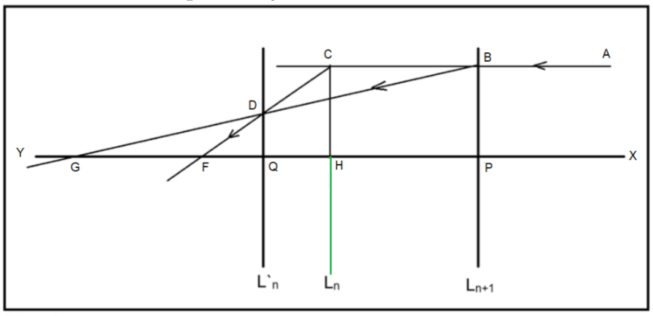

Diagram 2

$PQ = d_n$, $PG = f_{n+1}$ and $HF = f_n$.

Let $HQ = x$.

DQF and CHF are similar triangles. Hence: $\frac{CH}{DQ} = \frac{HF}{QF} \Rightarrow CH = DQ\frac{HF}{QF}$

DQG and BPG are similar triangles. Hence: $\frac{BP}{DQ} = \frac{PG}{QG} \Rightarrow BP = DQ\frac{PG}{QG}$

Since $CH = BP \Rightarrow DQ\frac{HF}{QF} = DQ\frac{PG}{QG} \Rightarrow \frac{HF}{QF} = \frac{PG}{QG} \Rightarrow \frac{HF}{HF-HQ} = \frac{PG}{PG-PQ} \Rightarrow \frac{f_n}{f_n - x} = \frac{f_{n+1}}{f_{n+1} - d_n}$

$$\Rightarrow f_{n+1}(f_n - x) = f_n(f_{n+1} - d_n) \Rightarrow f_{n+1}x = f_n d_n \Rightarrow x = \frac{f_n d_n}{f_{n+1}} = \frac{k_{n+1} d_n}{k_n} = \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$$

Hence: in the system of n lenses, if L_n is replaced by $L`_n$ and L_{n+1}, satisfying the following conditions, then the object focal length of the new system of n+1 lenses is equal to the object focal length of the original system of n lenses.

- $L`_n$ and L_{n+1} are separated by a distance of d_n
- $k_n = k`_n + k_{n+1} - k`_n d_n k_{n+1}$
- $L`_n$ is positioned at a distance of $\frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$ from and to the left of where L_n was

Hence: the distance $d`_{n-1}$ (between $L_{n-1}$ and $L`_n$) is given by

$$d`_{n-1} = d_{n-1} - \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$$

$$\Rightarrow d_{n-1} = d`_{n-1} + \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$$

$$\Rightarrow d_{n-1} k_n = \left(d`_{n-1} + \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}\right)(k`_n + k_{n+1} - k`_n d_n k_{n+1})$$

$$= d`_{n-1}(k`_n + k_{n+1} - k`_n d_n k_{n+1}) + d_n k_{n+1}$$

$$= d_n k_{n+1} + d`_{n-1}(k`_n + k_{n+1}) - d`_{n-1} k`_n d_n k_{n+1}$$

Hence, if the following replacements are made on the right-hand side of the equation

$$K_n = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s = a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n; d_n=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

- k_n is replaced with $k`_n + k_{n+1} - k`_n d_n k_{n+1}$

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- $d_{n-1}k_n$ is replaced with $d_n k_{n+1} + d`_{n-1}(k`_n + k_{n+1}) - d`_{n-1}k`_n d_n k_{n+1}$

then we ought to get the expression for K_{n+1}. This expression for K_{n+1} would be the same as K_n; except that n has been incremented by 1. This is effectively the induction step in the Mathematical Induction used immediately below.

E. Proof of the Proposed Formula for K_n

The following formula for K_n will now be proved by Mathematical Induction.

$$K_n = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

For the base case, when n = 1,

$$K_1 = \sum_{m=0}^{0}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=1;d_1=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} = (-1)^0 \prod_{s=1}^{1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1=1;d_1=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]$$

$$= \left(\sum_{\substack{r_1=a_0+1 \\ 0=a_0<a_1=1;d_1=1}}^{a_1} k_{r_1}\right) d_{a_1} = \left(\sum_{\substack{r_1=1 \\ d_1=1}}^{1} k_{r_1}\right) d_1 = (k_1)(1) = k_1$$

Thus, the formula is trivially true when n = 1.

For the inductive step, assume that the formula is true for n a certain value of n ≥ 1.

In the below, the square brackets following an expression have the label $E_\#$ (to identify the expression) followed by the applicable constraints [of which "$0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = 1; d_1 = 1$" is assumed to be always present until L_n is replaced].

Let the value of K_n be v. That is: $v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$, where $0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = n; d_n = 1$

Splitting v as E_1 (the summand corresponding to m = 0) plus E_2 (the summand corresponding to m = n-1) plus E_3 (the remaining summands), gives

$$v = \prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_1; a_1 = n]$$

$$+ (-1)^{n-1} \prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_2; a_n = n]$$

$$+ \sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_3; a_{m+1} = n]$$

In E_3, separating out the combinations for a_m = n-1 (denoted by E_4) and for $a_m \le$ n-2 (denoted by E_5), gives

$$v = \prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_1; a_1 = n]$$

$$+ (-1)^{n-1} \prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_2; a_n = n]$$

$$+ \sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_4; a_m = n-1, a_{m+1} = n]$$

$$+ \sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_5; a_m \le n-2, a_{m+1} = n]$$

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Recall the condition: $0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = n$.
Therefore: $m = n\text{-}1 \Rightarrow a_r = r,\ \forall\, r \in [0,n]$.
In the summand of E_4, when $m = n\text{-}1$, E_2 is obtained. Hence, merging E_4 and E_2 gives E_6.

$$\therefore v = \prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_1; a_1 = n]$$
$$+\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_6; a_m = n-1, a_{m+1} = n]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_5; a_m \le n-2, a_{m+1} = n]$$

In E_5 if m were to be set equal to 0, the summand becomes $(-1)^0 \prod_{s=1}^{0+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]$, with $a_1 = n$ $= \left(\sum_{r_1=a_{1-1}+1}^{a_1} k_{r_1}\right) d_{a_1} = \left(\sum_{r_1=a_0+1}^{n} k_{r_1}\right) d_n = \sum_{r_1=1}^{n} k_{r_1}$, since $a_0 = 0$ and $d_n = 1$. This is the same as E_1.
Note that in E_5, the condition $a_m \le n-2$ is applicable only when $m \ge 1$. When $m = 0$, $a_m = 0$

Hence, merging E_5 and E_1 to give E_7 results in

$$v = \sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_6; a_m = n-1, a_{m+1} = n]$$
$$+\sum_{m=0}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_7; a_m \le n-2, a_{m+1} = n]$$

Re-writing E_6 and E_7 as E_8 and E_9, respectively, gives

$$v = \sum_{m=1}^{n-1}\left\{(-1)^m \left(\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right)\left(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m}\right) d_{n-1}k_n\right\}[E_8; a_m = n-1, a_{m+1} = n]$$
$$+\sum_{m=0}^{n-2}\left\{(-1)^m \{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\}\left(\sum_{r_{m+1}=a_m+1}^{n-1} k_{r_{m+1}} + k_n\right)\right\}[E_9; a_m \le n-2, a_{m+1} = n]$$

Note that, at this stage, the assumed constraints "$a_{m+1} = n$ and $d_n = 1$" have been incorporated; they are no longer needed or applicable. Then the assumed constraint that remains is "$0 = a_0 < a_1 < \ldots < a_m < n$".
In E_8, the constraints $a_m = n - 1$ and $a_{m+1} = n$ can and will be replaced with $a_{m-1} \le n - 2$.
Replacing the constraints gives

$$v = \sum_{m=1}^{n-1}\left\{(-1)^m \{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m}\right) d_{n-1}k_n\right\}[E_8; a_{m-1} \le n-2]$$
$$+\sum_{m=0}^{n-2}\left\{(-1)^m \{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\}\left(\sum_{r_{m+1}=a_m+1}^{n-1} k_{r_{m+1}} + k_n\right)\right\}[E_9; a_m \le n-2]$$

L_n is now replaced by $\grave{L}_n$ and L_{n+1}, to have the value v be unchanged.
That is: k_n is to be replaced with $\grave{k}_n + k_{n+1} - \grave{k}_n d_n k_{n+1}$, and $d_{n-1}k_n$ is to be replaced with $d_n k_{n+1} + \grave{d}_{n-1}(\grave{k}_n + k_{n+1}) - \grave{d}_{n-1}\grave{k}_n d_n k_{n+1}$.
Hence, E_8 and E_9 become E_{10} and E_{11}, respectively, as the following shows.

$$\therefore v = \sum_{m=1}^{n-1}\left\{(-1)^m \{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m}\right)(d_n k_{n+1} + \grave{d}_{n-1}(\grave{k}_n\right.$$
$$\left. + k_{n+1}) - \grave{d}_{n-1}\grave{k}_n d_n k_{n+1})\right\}[E_{10}; a_{m-1} \le n-2]$$

$$+\sum_{m=0}^{n-2}\left\{(-1)^m\{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_{m+1}=a_m+1}^{n-1}k_{r_{m+1}}+k'_n+k_{n+1}-k'_n d_n k_{n+1}\right)\right\}[E_{11};a_m\le n-2]$$

Splitting up E_{10} as $E_{12}+E_{13}+E_{14}$, splitting up E_{11} as $E_{15}+E_{16}$ and dropping the dashes in d_{n-1} and k_n, for convenience, gives

$$v=\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}\right)d_n k_{n+1}\right\}[E_{12};a_{m-1}\le n-2]$$

$$+\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}\right)d_{n-1}(k_n+k_{n+1})\right\}[E_{13};a_{m-1}\le n-2]$$

$$-\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}\right)d_{n-1}k_n d_n k_{n+1}\right\}[E_{14};a_{m-1}\le n-2]$$

$$+\sum_{m=0}^{n-2}\left\{(-1)^m\{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_{m+1}=a_m+1}^{n-1}k_{r_{m+1}}+k_n+k_{n+1}\right)\right\}[E_{15};a_m\le n-2]$$

$$-\sum_{m=0}^{n-2}\left\{(-1)^m\{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}(k_n d_n k_{n+1})\right\}[E_{16};a_m\le n-2]$$

In E_{16} the dummy variable m is dropped by 1 to give E_{20}. Also, E_{13} is re-written as E_{17}, E_{14} is re-written as E_{18}, and E_{15} is re-written as E_{19}. Hence, the following

$$\therefore v=\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}\right)d_n k_{n+1}\right\}[E_{12};a_{m-1}\le n-2]$$

$$+\sum_{m=1}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{17};a_{m-1}\le n-2,a_m=n-1,a_{m+1}=n+1,d_{n+1}=1]$$

$$-\sum_{m=1}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{18};a_{m-1}\le n-2,a_m=n-1,a_{m+1}=n,a_{m+2}=n+1,d_{n+1}=1]$$

$$+\sum_{m=0}^{n-2}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{19};a_m\le n-2,a_{m+1}=n+1,d_{n+1}=1]$$

$$+\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}(k_n d_n k_{n+1})\right\}[E_{20};a_{m-1}\le n-2]$$

From here on, the constraint $d_{n+1}=1$ is assumed.

E_{12} and E_{20} are now added together to give E_{21}, as the following shows: $E_{12}+E_{20}$

$$=\sum_{m=1}^{n-1}\left\{\left((-1)^m\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}\right)d_n k_{n+1}+k_n d_n k_{n+1}\right)\right\}[E_{21};a_{m-1}\le n-2]$$

$$=\sum_{m=1}^{n-1}\left\{\left((-1)^m\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m}+k_n\right)d_n k_{n+1}\right)\right\}[E_{21};a_{m-1}\le n-2]$$

$$=\sum_{m=1}^{n-1}\left\{\left((-1)^m\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n}k_{r_m}\right)d_n k_{n+1}\right)\right\}[E_{21};a_{m-1}\le n-2]$$

$$=\sum_{m=1}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{21};a_{m-1}\le n-2,a_m=n,a_{m+1}=n+1]$$

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Note that the assumed constraint "$a_m < n$" is no longer applicable and is replaced with "$a_m \leq n$".

$$\text{Hence, } v = \sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{17}; a_{m-1} \leq n-2, a_m = n-1, a_{m+1} = n+1]$$
$$-\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{18}; a_{m-1} \leq n-2, a_m = n-1, a_{m+1} = n, a_{m+2} = n+1]$$
$$+\sum_{m=0}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{19}; a_m \leq n-2, a_{m+1} = n+1]$$
$$+\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{21}; a_{m-1} \leq n-2, a_m = n, a_{m+1} = n+1]$$

The following are now being performed.
E_{17} is split as E_{22} (the summand corresponding to m = n-1) plus E_{23} (the remaining summands).
The dummy variable m in E_{18} is dropped by 1 to give E_{24}.
E_{19} is split as E_{25} (the summand corresponding to m = 0) plus E_{26} (the remaining summands).

$$\text{Hence, } v = \sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{21}; a_{m-1} \leq n-2, a_m = n, a_{m+1} = n+1]$$
$$+(-1)^{n-1}\prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_{22}; a_{n-2} \leq n-2, a_{n-1} = n-1, a_n = n+1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{23}; a_{m-1} \leq n-2, a_m = n-1, a_{m+1} = n+1]$$
$$+\sum_{m=2}^{n}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{24}; a_{m-2} \leq n-2, a_{m-1} = n-1, a_m = n, a_{m+1} = n+1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_{25}; a_1 = n+1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{26}; a_m \leq n-2, a_{m+1} = n+1]$$

The following are now being performed.
E_{21} is split as E_{27} (the summand corresponding to m = 1) plus E_{28} (the remaining summands).
E_{24} is split as E_{30} (the summand corresponding to m = n) plus E_{31} (the remaining summands).
E_{23} (with a_m = n-1) and E_{26} (with $a_m \leq$ n-2) are merged together to give E_{29} (with $a_m \leq$ n-1). Note that this merger is possible because the constraint "$a_{m+1} = n + 1$" is common to both and also in E_{23} "$a_{m-1} \leq n-2$" is effective. If in E_{23}, a_{m-1} has a smaller maximum value, then this merger would not be possible.

$$\text{Hence, } v = -\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_{27}; a_1 = n, a_2 = n+1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{28}; a_{m-1} \leq n-2, a_m = n, a_{m+1} = n+1]$$
$$+(-1)^{n-1}\prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_{22}; a_{n-2} \leq n-2, a_{n-1} = n-1, a_n = n+1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\} [E_{29}; a_m \leq n-1, a_{m+1} = n+1]$$

$$+(-1)^n\prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{30};a_{n-2}\leq n-2,a_{n-1}=n-1,a_n=n,a_{n+1}=n+1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{31};a_{m-2}\leq n-2,a_{m-1}=n-1,a_m=n,a_{m+1}=n+1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{25};a_1=n+1]$$

E_{28} (with $a_{m-1} \leq n-2$) and E_{31} (with $a_{m-1} = n-1$) are merged to give E_{32} (with $a_{m-1} \leq n-1$).
In the summand of E_{29}, when m = n-1, E_{22} is obtained. Therefore, E_{29} and E_{22} are merged to give E_{33}.

$$\therefore v=-\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{27};a_1=n,a_2=n+1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{32};a_{m-1}\leq n-1,a_m=n,a_{m+1}=n+1]$$
$$+\sum_{m=1}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{33};a_m\leq n-1,a_{m+1}=n+1]$$
$$+(-1)^n\prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{30};a_{n-2}\leq n-2,a_{n-1}=n-1,a_n=n,a_{n+1}=n+1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{25};a_1=n+1]$$

E_{33} is split as E_{34} (the summand corresponding to m = 1) plus E_{35} (the remaining summands), resulting in

$$v=-\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{27};a_1=n,a_2=n+1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{32};a_{m-1}\leq n-1,a_m=n,a_{m+1}=n+1]$$
$$-\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{34};a_1\leq n-1,a_2=n+1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{35};a_m\leq n-1,a_{m+1}=n+1]$$
$$+(-1)^n\prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{30};a_{n-2}\leq n-2,a_{n-1}=n-1,a_n=n,a_{n+1}=n+1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{25};a_1=n+1]$$

E_{27} (with $a_1 = n$) and E_{34} (with $a_1 \leq n-1$) are merged to give E_{36} (with $a_1 \leq n$).
E_{32} (with $a_m = n$) and E_{35} (with $a_m \leq n-1$) are merged to give E_{37} (with $a_m \leq n$).

$$\therefore v=-\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{36};a_1\leq n,a_2=n+1]$$

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$$+\sum_{m=2}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_{37};a_m\leq n,a_{m+1}=n+1]$$

$$+(-1)^n\prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{30};a_{n-2}\leq n-2,a_{n-1}=n-1,a_n=n,a_{n+1}=n+1]$$

$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_{25};a_1=n+1]$$

Note that the assumed constraint of “$0 = a_0 < a_1 < \ldots < a_m \leq n$” and “$d_{n+1} = 1$” together with “$a_{m+1} = n+1$” (from E_{37}) can be merged into “$0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = n+1, d_{n+1} = 1$”

In the summand of E_{37}, when m = 0, 1, and n, the following are obtained: E_{25}, E_{36}, and E_{30}, respectively.

$$\therefore v=\sum_{m=0}^{n}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[0=a_0<a_1<\cdots<a_m<a_{m+1}=n+1,d_{n+1}=1]$$

$$=\sum_{m=0}^{n}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1\\0=a_0<a_1<\cdots<a_m<a_{m+1}=n+1,d_{n+1}=1}}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}$$

This expression for v is the same as for K_n, except that n has been replaced with n+1. This concludes the proof of the formula for the Object Focal Length of a system of n lenses.

F. Proof that Object Power Equals Image Power

The object power of a system of n lenses is given by

$$K_n(k_1,d_1,k_2,d_2,\ldots,k_{n-1},d_{n-1},k_n)=\sum_{m=0}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1\\0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}$$

By interchanging d_r and d_{n-r}, ∀ r ∈ [1,n-1], and by interchanging k_r and k_{n+1-r}, ∀ r ∈ [1,n], in the expression for $K_n(k_1, d_1, k_2, d_2 \ldots, k_n)$, we will get the formula for the image power, say v.
Note: to simplify the algebra, d_0 (just like d_n) is defined to be 1; and the interchanging of d_r and d_{n-r} will be done ∀ r ∈ [1,n], instead of ∀ r ∈ [1,n-1]. Therefore, the image power of the system is given by $v = K_n(k_n, d_{n-1}, k_{n-1}, \ldots, d_2, k_2, d_1, k_1)$.

$$\therefore v=\sum_{m=0}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1\\0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s}k_{n+1-r_s}\right)d_{n-a_s}\right]\right\}$$

In the expression for v, changing dummy variables from the r’s to t’s via $t_{m+2-s} = n+1-r_s$ gives

$$v=\sum_{m=0}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{\substack{t_{m+2-s}=n+1-a_s\\0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{n-a_{s-1}}k_{t_{m+2-s}}\right)d_{n-a_s}\right]\right\}$$

Changing variables from the a’s to b’s via $b_{m+1-s} = n-a_s$ gives

$$v=\sum_{m=0}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{\substack{t_{m+2-s}=b_{m+1-s}+1\\0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_{m+2-s}}k_{t_{m+2-s}}\right)d_{b_{m+1-s}}\right]\right\}$$

Changing the dummy variable from s to x via x = m+2-s gives

$$v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right) d_{b_{x-1}}\right]\right\}$$

$$= \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right)\right]\left[d_{b_0} d_{b_1} \ldots . d_{b_{m-1}} d_{b_m}\right]\right\}$$

Now, $b_0 = 0$ and $d_0 = 1$. $\therefore d_{b_0} = 1$

Also, $b_{m+1} = n$ and $d_n = 1$. $\therefore d_{b_{m+1}} = 1$

$$\therefore v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right)\right]\left[d_{b_1} \ldots . d_{b_{m-1}} d_{b_m} d_{b_{m+1}}\right]\right\}$$

$= \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right) d_{b_x}\right]\right\}$; i.e. the **object power**

Thus: image power = object power, and this common value is called the **power** of the system.

Hence: $K_n(k_n, d_{n-1}, k_{n-1}, \ldots, d_2, k_2, d_1, k_1) = K_n(k_1, d_1, k_2, d_2, \ldots, k_{n-1}, d_{n-1}, k_n)$; with $d_0 = d_n = 1$.

That is, if the first lens and the last lens were to interchange positions, the second lens and the second to last lens were to interchange positions, etc., then the power (and focal length) of the system remains unchanged.

IV. GENERALIZED FORMULAS FOR h_1 AND h_2

A. Notation Used

The diagram below shows a system of n lenses.

The following notations are used

- the image focus is denoted by $I_{n,2}$
- the second principal point is denoted by $H_{n,2}$
- the focal length is F_n (the distance $I_{n,2}H_{n,2}$)
- the distance between L_n and $H_{n,2}$ (i.e. $DH_{n,2}$) is denoted by $h_{n,2}$
- the distance between L_1 and $H_{n,1}$ (the first principal point; not shown in the diagram) is denoted by $h_{n,1}$

Note that $h_{n,1}$ and $h_{n,2}$ are the generalized versions of h_1 and h_2, respectively.

With L_n being absent, i.e. we are dealing with a system of n-1 lenses, the corresponding points and lengths are denoted by n being replaced with n-1.

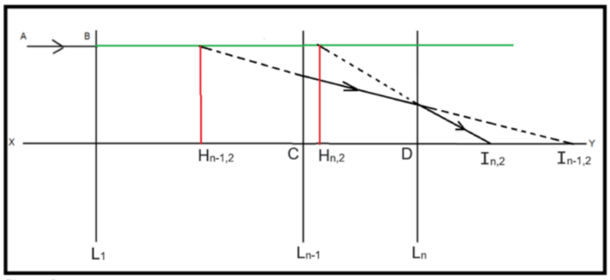

Diagram 3

B. The Formulas

Without L_n, an infinitely distant object has its image at $I_{n-1,2}$. Therefore, a virtual object at $I_{n-1,2}$ will have its real image (under refraction by L_n acting alone) at $I_{n,2}$.

Thus:

- the object distance, $u = -(DI_{n-1,2}) = -(H_{n-1,2}I_{n-1,2} - H_{n-1,2}D) = -(H_{n-1,2}I_{n-1,2} - \{H_{n-1,2}C + CD\}) = -(F_{n-1} - h_{n-1,2} - d_{n-1})$
- the image distance, $v = DI_{n,2} = H_{n,2}I_{n,2} - H_{n,2}D = F_n - h_{n,2}$

Using $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ (with L_n acting alone), gives

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$$\frac{1}{-(F_{n-1}-h_{n-1,2}-d_{n-1})}+\frac{1}{F_n-h_{n,2}}=\frac{1}{f_n}\Rightarrow\frac{1}{F_n-h_{n,2}}=\frac{1}{f_n}+\frac{1}{F_{n-1}-h_{n-1,2}-d_{n-1}}=\frac{1}{f_n}-\frac{1}{d_{n-1}-(F_{n-1}-h_{n-1,2})}$$

$\Rightarrow F_n-h_{n,2}=\dfrac{1}{\frac{1}{f_n}-\frac{1}{d_{n-1}-(F_{n-1}-h_{n-1,2})}}$, a recurrence relation for $F_n-h_{n,2}$ ------- (1)

$$\Rightarrow h_{n,2}=F_n-\frac{1}{\frac{1}{f_n}-\frac{1}{d_{n-1}-(F_{n-1}-h_{n-1,2})}}=F_n-\frac{1}{\frac{1}{f_n}-\frac{1}{d_{n-1}-\frac{1}{\frac{1}{f_{n-1}}-\frac{1}{d_{n-2}-(F_{n-2}-h_{n-2,2})}}}}$$ [using equation (1)]

⋮

$$=F_n-\cfrac{1}{\cfrac{1}{f_n}-\cfrac{1}{d_{n-1}-\cfrac{1}{\cfrac{1}{f_{n-1}}-\cfrac{1}{\ddots\,\cfrac{1}{d_2-\cfrac{1}{\cfrac{1}{f_2}-\cfrac{1}{d_1-f_1}}}}}}}$$

[Note: for a system consisting of a single lens, $h_{1,1}=h_{1,2}=0$.]

Hence: $$h_{n,2}=F_n-\cfrac{1}{\cfrac{1}{f_n}-\cfrac{1}{d_{n-1}-\cfrac{1}{\cfrac{1}{f_{n-1}}-\cfrac{1}{\ddots\,\cfrac{1}{d_2-\cfrac{1}{\cfrac{1}{f_2}-\cfrac{1}{d_1-f_1}}}}}}}$$

By interchanging d_r and d_{n-r} and interchanging f_r and f_{n+1-r}, $\forall\, r \in [1,n]$, in the expression for $h_{n,2}$, we will get the formula for $h_{n,1}$.

Hence: $$h_{n,1}=F_n-\cfrac{1}{\cfrac{1}{f_1}-\cfrac{1}{d_1-\cfrac{1}{\cfrac{1}{f_2}-\cfrac{1}{\ddots\,\cfrac{1}{d_{n-2}-\cfrac{1}{\cfrac{1}{f_{n-1}}-\cfrac{1}{d_{n-1}-f_n}}}}}}}$$

V. GENERALIZED GAUSSIAN LENS EQUATION

A. Notation Used

Let the distance between

- an object and the first lens be u
- the image and the last lens be v
- the first lens and the first principal point be $h_{n,1}$
- the last lens and the second principal point be $h_{n,2}$

Note that for a system of 2 lenses (n = 2), $h_{n,1}$ is the same as h_1, and $h_{n,2}$ is the same as h_2.

B. Lens Equation

The generalized Gaussian lens equation is $\frac{1}{u+h_{n,1}}+\frac{1}{v+h_{n,2}}=\frac{1}{F_n}$.

This formula will be proved by mathematical induction; with the last lens L_n being replaced with $L`_n$ and L_{n+1}.

The result is true for n = 1 (where $h_{1,1}=h_{1,2}=0$).

It is also true when n = 2 (a standard result).

The diagram below shows an object O and its image I formed by a system of lenses.

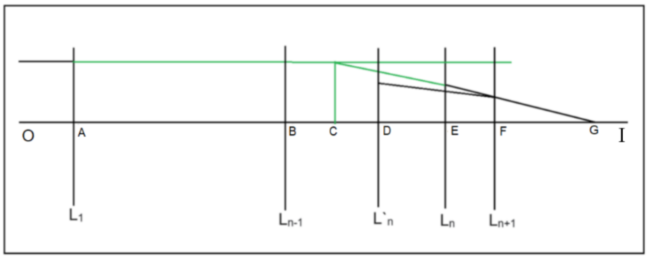

Diagram 4

Initially, there were n lenses; $L^{`}_{n}$ and L_{n+1} were not present.
L_n is then replaced with an **equivalent** system of 2 lenses, $L^{`}_{n}$ and L_{n+1}, meaning that

- if $L^{`}_{n}$ and $L_{n+1}$ are separated by a distance of $d_n$ apart, and $k_n$, $k^{`}_{n}$, and k_{n+1} are the power of L_n, $L^{`}_{n}$, and L_{n+1}, respectively, then
  - $k_n = k^{`}_{n} + k_{n+1} - k^{`}_{n} d_n k_{n+1}$; or in terms of focal length $\frac{1}{f_n} = \frac{1}{f^{`}_{n}} + \frac{1}{f_{n+1}} - \frac{d_n}{f^{`}_{n} f_{n+1}}$

    That is $f_n = \frac{f^{`}_{n} f_{n+1}}{f^{`}_{n} + f_{n+1} - d_n}$ ------- (2)
  - $L^{`}_{n}$ is positioned at a distance of $\frac{d_n k_{n+1}}{k^{`}_{n} + k_{n+1} - k^{`}_{n} d_n k_{n+1}}$ (or $\frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$) from and to the left of where L_n was

    That is DE = $\frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ ------- (3)
- the focal length of the new system of n+1 lenses is equal to the focal length of the old system of n lenses
- the position of the image focus G remains unchanged
- the position of the image I remains unchanged

The idea of the above is to simplify the algebra required to show that if
"$\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$" is true for a specific value of n, say when n = c, then the formula is also true when n = c+1.
The following notations are used (before the replacement of L_n with $L^{`}_{n}$ and L_{n+1})

- The distance between
  - object and L_1 (i.e. OA) = u
  - image and L_n (i.e. EI) = v
  - first principal point and L_1 = $h_{n,1}$
  - second principal point and L_n (i.e. CE) = $h_{n,2}$
  - L_{n-1} and L_n (i.e. BE) = d_{n-1}
- The focal length of the system (i.e. CG) = F_n

The following notations are used (after the replacement of L_n with $L^{`}_{n}$ and L_{n+1})

- The distance between
  - object and L_1 (i.e. OA) = u
  - image and L_{n+1} (i.e. FI) = v`

    v` = FI = EI – EF = EI – (DF – DE) = $v - d_n + \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ [using equation (3)]

    That is $v = v^{`} + d_n - \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ ------- (4)
  - first principal point and L_1 = $h^{`}_{n+1,1}$
  - second principal point and L_{n+1} (i.e. CF) = $h^{`}_{n+1,2}$

    $h^{`}_{n+1,2}$ = CF = CE + EF = CE + (DF – DE) = $h_{n,2} + d_n - \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ [using equation (3)]

    That is $h^{`}_{n+1,2} = h_{n,2} + d_n - \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ ------- (5)
  - L_{n-1} and $L^{`}_{n}$ (i.e. BD) = $d^{`}_{n-1}$

    $d^{`}_{n-1}$ = BD = BE – DE = $d_{n-1} - \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ [using equation (3)]

    That is $d_{n-1} = d^{`}_{n-1} + \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}$ ------- (6)
- The focal length of the system (i.e. CG) = $F^{`}_{n+1}$ = F_n

With the above arrangement in place, it will now be shown that $h_{n,1} = h^{`}_{n+1,1}$

$$h_{n,1} = F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d_{n-1} - f_n}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\left(d^{`}_{n-1} + \frac{d_n f^{`}_{n}}{f^{`}_{n} + f_{n+1} - d_n}\right) - \left(\frac{f^{`}_{n} f_{n+1}}{f^{`}_{n} + f_{n+1} - d_n}\right)}}}}}}}$$ [using equation (6) and equation (2)]

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$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots - \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d^{`}_{n-1} + f^{`}_n\left(\frac{d_n - f_{n+1}}{f^{`}_n + f_{n+1} - d_n}\right)}}}}}}} = F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots - \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d^{`}_{n-1} + f^{`}_n\left(\cfrac{1}{\cfrac{f^{`}_n}{d_n - f_{n+1}} - 1}\right)}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots - \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d^{`}_{n-1} - f^{`}_n\left(\cfrac{1}{1 - \cfrac{f^{`}_n}{d_n - f_{n+1}}}\right)}}}}}}} = F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots - \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d^{`}_{n-1} - f^{`}_n\left(\cfrac{1}{\cfrac{f^{`}_n}{f^{`}_n} - \cfrac{f^{`}_n}{d_n - f_{n+1}}}\right)}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots - \cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d^{`}_{n-1} - \cfrac{1}{\cfrac{1}{f^{`}_n} - \cfrac{1}{d_n - f_{n+1}}}}}}}}}} = h^{`}_{n+1,1}$$

The fact that $h_{n,1} = h^{`}_{n+1,1}$ should not be surprising, since the object focus and the (object) focal length remain unchanged when $L_n$ is replaced with an equivalent system of two lenses, $L^{`}_n$ and L_{n+1}.

Now, for the inductive step of the Mathematical Induction, assume that $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$.

Since $h_{n,1} = h^{`}_{n+1,1}$, $v = v^{`} + d_n - \frac{d_n f^{`}_n}{f^{`}_n + f_{n+1} - d_n}$ [equation (4)], and $F_n = F^{`}_{n+1}$, this implies that

$$\frac{1}{u + h^{`}_{n+1,1}} + \frac{1}{v^{`} + d_n - \frac{d_n f^{`}_n}{f^{`}_n + f_{n+1} - d_n} + h_{n,2}} = \frac{1}{F^{`}_{n+1}}$$

$$\therefore \frac{1}{u + h^{`}_{n+1,1}} + \frac{1}{v^{`} + h^{`}_{n+1,2}} = \frac{1}{F^{`}_{n+1}}, \text{ since } h^{`}_{n+1,2} = h_{n,2} + d_n - \frac{d_n f^{`}_n}{f^{`}_n + f_{n+1} - d_n} \text{ [equation (5)]}$$

Dropping the dashes, for convenience, gives $\frac{1}{u + h_{n+1,1}} + \frac{1}{v + h_{n+1,2}} = \frac{1}{F_{n+1}}$; i.e. the same assumed formula except that n is replaced with n+1.

Thus: the formula $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$ is true for all positive integer values of n.

VI. OTHER FORMULAS

In this section, the following is proved: $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

The diagram below shows a ray of light, parallel to the principal axis of a system of n+1 lenses, being refracted by the system to go through H, the image focus of the system. Without L_{n+1}, the ray would have gone through I, the image focus of the system of the preceding n lenses.

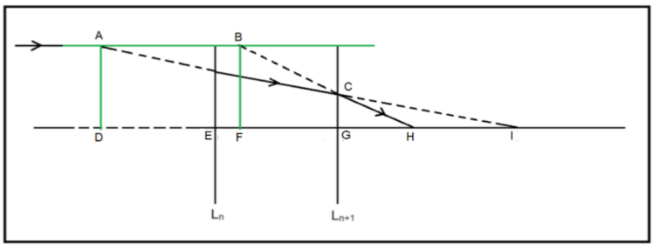

Diagram 5

BFH and CGH are similar triangles. Hence: $\frac{BF}{CG} = \frac{FH}{GH} = \frac{FH}{FH - FG} = \frac{F_{n+1}}{F_{n+1} - h_{n+1,2}}$

ADI and CGI are similar triangles. Hence: $\frac{AD}{CG} = \frac{DI}{GI} = \frac{DI}{DI - DG} = \frac{DI}{DI-(DE+EG)} = \frac{F_n}{F_n - h_{n,2} - d_n}$

BF = AD. Hence: $\frac{BF}{CG} = \frac{AD}{CG} \Rightarrow \frac{F_{n+1}}{F_{n+1} - h_{n+1,2}} = \frac{F_n}{F_n - h_{n,2} - d_n}$

$\Rightarrow \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{F_n}{F_{n+1}}$ ------- (7)

Continuing: $(F_n - h_{n,2} - d_n)F_{n+1} = (F_{n+1} - h_{n+1,2})F_n \Rightarrow (h_{n,2} + d_n)F_{n+1} = h_{n+1,2}F_n$

$\Rightarrow \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{F_n}{F_{n+1}}$ ------- (8)

Recall equation (1): $F_n - h_{n,2} = \frac{1}{\frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})}}$

$\Rightarrow \frac{1}{F_n - h_{n,2}} = \frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})} \Rightarrow \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})} = \frac{1}{f_n} - \frac{1}{F_n - h_{n,2}} = \frac{F_n - h_{n,2} - f_n}{f_n(F_n - h_{n,2})}$

$\Rightarrow \frac{d_{n-1} - F_{n-1} + h_{n-1,2}}{F_n - h_{n,2}} = \frac{f_n}{F_n - h_{n,2} - f_n} \Rightarrow$ (by increasing n by 1) $\frac{d_n - F_n + h_{n,2}}{F_{n+1} - h_{n+1,2}} = \frac{f_{n+1}}{F_{n+1} - h_{n+1,2} - f_{n+1}}$

$\Rightarrow \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$ ------ (9)

Hence, from equation (7), equation (8), and equation (9): $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

In the above result, if d_r and d_{n+1-r} are interchanged ∀ r ∈ [1,n], and if f_r and f_{n+2-r} are interchanged ∀ r ∈ [1,n+1], then $h_{n,2}$ will have to be replaced with $h_{n,1}$ and $h_{n+1,2}$ will have to be replaced with $h_{n+1,1}$

Hence: $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,1} - d_1}{F_{n+1} - h_{n+1,1}} = \frac{h_{n,1} + d_1}{h_{n+1,1}} = \frac{f_1}{f_1 + h_{n+1,1} - F_{n+1}}$, where

- F_n is the focal length of the system of lenses where the L_1 is absent
- $h_{n,1}$ is the distance between the first principal point and L_2 (with L_1 being absent)

Note the following (which will be used in the next section):

- $\frac{d_n + h_{n,2} - F_n}{F_n} = \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}}$ ------- (10)
- $\frac{f_{n+1} + h_{n+1,2} - F_{n+1}}{f_{n+1}F_{n+1}} = \frac{1}{F_n}$ ------- (11)

VII. TRANSVERSE MAGNIFICATION FORMULA

The transverse magnification M_n of a system of n lenses is given by $M_n = \frac{v + h_{n,2}}{F_n} - 1$.

This will be proved by the method of Induction.

The formula is trivially true when n = 1, since $h_{1,2} = 0$.

Also, the formula is a standard one when n = 2.

For the inductive step, assume that $M_n = \frac{v + h_{n,2}}{F_n} - 1$ and consider the following situation.

The diagram below shows an image I_{n+1} produced by an object O under the effect of a system of n+1 lenses (only the last 2 lenses are shown).

Without L_{n+1}, O would have produced the image I_n, and in this case, the image distance v = AI_n in the assumed formula $M_n = \frac{v + h_{n,2}}{F_n} - 1$.

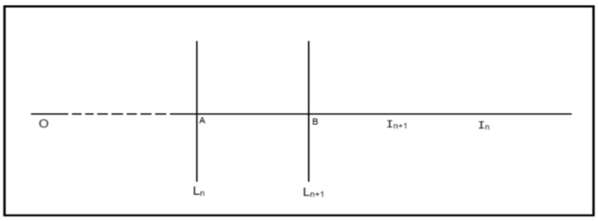

Diagram 6

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A virtual object I_n produces a real image I_{n+1} under refraction by L_{n+1} (alone)
For this single lens case,

- the object distance, $u = -BI_n$ (the negative sign is because of the **virtual** object) $= -(AI_n - AB) = -(v - d_n)$
- The image distance $v' = BI_{n+1}$

Using "$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$" gives $\frac{1}{-(v - d_n)} + \frac{1}{v'} = \frac{1}{f_{n+1}} \Rightarrow \frac{1}{(v - d_n)} = \frac{1}{v'} - \frac{1}{f_{n+1}} = \frac{f_{n+1} - v'}{v' f_{n+1}} \Rightarrow v - d_n = \frac{v' f_{n+1}}{f_{n+1} - v'}$

$\Rightarrow v = d_n + \frac{v' f_{n+1}}{f_{n+1} - v'}$ ------- (12)

Since magnification is multiplicative, it means that $M_{n+1} = M_n\left(-\{\frac{v'}{f_{n+1}} - 1\}\right)$; note the negative sign because of the **virtual** object.

$$\therefore M_{n+1} = -\left(\frac{v + h_{n,2}}{F_n} - 1\right)\left(\frac{v'}{f_{n+1}} - 1\right) = -\left(\frac{\left\{d_n + \frac{v' f_{n+1}}{f_{n+1} - v'}\right\} + h_{n,2}}{F_n} - 1\right)\left(\frac{v'}{f_{n+1}} - 1\right) \text{ [using equation (12)]}$$

$$= -\left(\frac{\frac{\{d_n + h_{n,2}\}\{f_{n+1} - v'\} + v' f_{n+1}}{(f_{n+1} - v')}}{F_n} - 1\right)\left(\frac{v' - f_{n+1}}{f_{n+1}}\right) = -\left(\frac{\{d_n + h_{n,2}\}\{f_{n+1} - v'\} + v' f_{n+1} - (f_{n+1} - v')F_n}{(f_{n+1} - v')F_n}\right)\left(\frac{v' - f_{n+1}}{f_{n+1}}\right)$$

$$= \frac{\{d_n + h_{n,2} - F_n\}\{f_{n+1} - v'\} + v' f_{n+1}}{f_{n+1} F_n} = \frac{\{d_n + h_{n,2} - F_n\}\{f_{n+1} - v'\}}{f_{n+1} F_n} + \frac{v'}{F_n} = \left(\frac{d_n + h_{n,2} - F_n}{F_n}\right)\left(1 - \frac{v'}{f_{n+1}}\right) + \frac{v'}{F_n}$$

$$= \left(\frac{h_{n+1,2} - F_{n+1}}{F_{n+1}}\right)\left(1 - \frac{v'}{f_{n+1}}\right) + \frac{v'}{F_n} \text{ [using equation (10)]} = \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{(h_{n+1,2} - F_{n+1})v'}{f_{n+1} F_{n+1}} + \frac{v'}{F_n}$$

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{(h_{n+1,2} - F_{n+1})v'}{f_{n+1} F_{n+1}} - \frac{v'}{F_{n+1}} + \frac{v'}{F_{n+1}} + \frac{v'}{F_n} \text{ [adding and subtracting } \frac{v'}{F_{n+1}}\text{]}$$

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{(h_{n+1,2} - F_{n+1} + f_{n+1})v'}{f_{n+1} F_{n+1}} + \frac{v'}{F_{n+1}} + \frac{v'}{F_n} = \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{v'}{F_n} + \frac{v'}{F_{n+1}} + \frac{v'}{F_n} \text{ [using equation (11)]}$$

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} + \frac{v'}{F_{n+1}} = \frac{v' + h_{n+1,2}}{F_{n+1}} - 1$$

Dropping the dash, for convenience, gives $M_{n+1} = \frac{v + h_{n+1,2}}{F_{n+1}} - 1$. This is the same formula for M_n, except that n is replaced with n+1. The induction step is completed.

VIII. CONCLUSION

In a system of n thin coaxial lenses let

- the distance between
  - an object and the first lens be u
  - the image and the last lens be v
  - the first lens and the first principal point be $h_{n,1}$
  - the last lens and the second principal point be $h_{n,2}$
  - the distance between the r^{th} lens and $(r+1)^{th}$ lens be d_r
- the focal length of
  - the r^{th} lens be f_r
  - the system be F_n
- the transverse magnification be m

The following formulas are valid

- $\frac{1}{F_n} = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s = a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;\, d_n=1}}^{a_s} \frac{1}{f_{r_s}}\right) d_{a_s}\right]\right\}$, where, for a fixed value of m, all combinations of values of the a's (satisfying the condition "$0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = n$") are taken in the inner sum.
- $h_{n,2} = F_n - \cfrac{1}{\cfrac{1}{f_n} - \cfrac{1}{d_{n-1} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_2 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{d_1 - f_1}}}}}}}}$
- $h_{n,1} = F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d_{n-1} - f_n}}}}}}}}$

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- $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$
- $M_n = \frac{v + h_{n,2}}{F_n} - 1$
- $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

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