Calculation of the TOV Limit Based on Neutron Degeneracy Pressure

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***Abstract:** Original theory on the mass limit beyond which a cold, non-rotating neutron star cannot be formed, instead only stellar black holes will be created, was stipulated by J.R. Oppenheimer and G.M. Volkoff based on R.C. Tolman's work in 1939. The limit calculated from the equation established by them is known as the TOV limit which is analogous to the Chandrasekhar limit for White Dwarfs. But the results obtained using the formula was found to be not valid today. Subsequent theoretical works place the limit in the range 1.5 to 3 solar masses. There are several basic theories and related formulae for calculating the TOV limit. In this article a different and novel approach has been adopted to calculate the TOV limit using the theory on neutron degeneracy pressure. As per present calculations, the TOV limit is around 2.928 times the solar mass. These calculations also highlight two aspects which are conceptually new; first, a black hole having mass higher than the TOV limit can also become a neutron star and both can coexist concurrently up to a certain limit; and second, that upper limit of star mass beyond which a black hole will explode in supernova before becoming a neutron star is 7.15 times the solar mass as at that stage the gravitational energy of the black hole will be equal or exceed to its nuclear binding energy.*

***Keywords.** Electron Degeneracy Pressure, Neutron Degeneracy Pressure, Escape Velocity, the Chandrasekhar Limit, Nuclear Binding Energy.*

I. INTRODUCTION

Neutron stars are formed after the supernova explosions of massive cold or dead stars having masses around 7 and more solar mass. Due to the explosion the outer crust of a cold star disintegrates and gets scattered in the space in the form of cosmic dusts that create nebula. Its hugely dense smaller inner core continues to collapse under the influence of the gravitational pressure of the star. Consequently neutron degeneracy pressure (Physics Forum [1]) starts developing inside the star at its sub-atomic level following the Pauli's Exclusion Principle which acts outwardly and counters its gravitational pressure. At a particular radius, when both these two pressures become equal, a stable neutron star (earthsky.org [2]) is formed. While the gravitational pressure is developed obeying the laws of classical physics, the neutron degeneracy pressure is a type of quantum pressure which is analogous to the electron degeneracy pressure that is responsible for formation of white dwarfs.

Neutron degeneracy pressure acts outwardly and counters the inwardly acting gravitational pressure of a cold star. At the TOV limit, that is analogous to the Chandrasekhar limit (sciencedirect.com [3]) for white dwarfs, following two conditions of the cold star are concurrently satisfied:

(1) The gravitational pressure and the neutron degeneracy pressure (NDP) will be equal.
(2) The Escape Velocity of the star will be equal to the velocity of light.

The equation that determines the TOV limit has been formulated on these two criteria mentioned above. The TOV limit is presently considered to be the maximum mass a neutron star can support. It is also a lower limit of mass of a stellar black hole. However, detailed calculations based on the neutron degeneracy pressure suggest the following.

The TOV limit calculated based on the equation derived in this paper is around **2.928** times the solar mass. Theoretically, the TOV limit is the least mass of a stellar black hole. At and below this mass limit, only neutron stars are formed.

A. Radius of the cold star at TOV limit is its Schwarzschild radius

In contrary to the common belief, the calculations reveal that neutron stars can also be formed beyond the TOV limit. The stellar black holes formed up to a certain mass higher than the TOV limit will also become neutron stars at reduced respective radius. Basic difference is the neutron stars having masses below the TOV limit are visible but the neutron stars having masses equal to and higher than the TOV limit are totally indistinguishable as those are black holes too. Though theoretically visible, even the neutron stars below the TOV limit are extremely difficult to detect as those are located at great distances and have minuscule volumes.

However, there is an upper limit of mass up to which a black hole will become a neutron star. As per the present calculations, this limit for a non-rotating and constant-density black hole is 7.15 times of solar mass. Beyond the mass limit of (around) **7.15 MΘ**, a black hole, prior to becoming a neutron star, will explode in supernova as at that stage the gravitational energy of the black hole will equal or surpass to its nuclear binding energy.

II. CALCULATION OF THE TOV LIMIT

A. The TOV limit has Been Derived from the Following Equations.

(1) Gravitational Pressure (Gr. Pr) = Neutron Degeneracy Pressure (NDP) (J Timilin 2013 [4])

Gr.Pr.= (1/5)* (G*M^2)*[{(4/3)*(π)}^(1/3)]*{(V)^-(4/3)} (kg.m^-2)

Manuscript received on 01April 2024 | Revised Manuscript received on 12 April 2024 | Manuscript Accepted on 15 April 2024 | Manuscript published on 30 June 2024.

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NDP = {(π^3*ħ^2)/(15*Mn)}*[{(3*Nn)/(π*V)}^(5/3)] (kg.m^-2)

Where,

G = Universal Gravitational Constant = 6.67E-11 m^3.kg^-1.s^-2

M = Mass of the cold star in kg.

MΘ = Solar mass, 1.989E+30 kg

V = Volume of the star having radius R (in m), m^3

ħ = reduced Planck Constant = 1.055E-34 J.s

Mn = Mass of neutron = 1.675E-27 kg

Nn = Number of neutron in the star

R = Radius of the star in m

c = Velocity of light = 3E+8 m.s^-1

(2) Escape Velocity, E.V., **(2*G*M/R)^(1/2) = c,**

Equating the gravitational pressure and the NDP and using the relationship between mass (M) and radius (R) derived from the E.V, we get a precise formula for the TOV Limit,

M (TOV Limit) = (8.216*10^12)*{ħ^2*MN^-(8/3)*G^-2}^(3/4) kg

= 5.825*10^30 kg

= 2.9287 MΘ. **(Approx.)**

III. CONCLUSIONS

According to the prevailing concept, the TOV Limit calculated as per the aforementioned formulae signifies that 2.928 times the solar mass is the maximum mass a neutron star can have. Beyond this limit, only (stellar) black holes can be formed, not neutron stars.

However, detailed calculations suggest that the cold stars having mass beyond the TOV limit will become black holes first and then, as their NDPs are less than the corresponding gravitational pressures, will continue to collapse further and eventually those black holes will become neutron stars too. But being tiny black holes (radii less than 10 kilometers and the nearest being at a distance far outside our solar system somewhere in the Milky Way), these neutron stars are practically invisible and undetectable even by most powerful telescopes.

Also, there is a maximum limit of mass up to which such invisible neutron stars can be formed from the black holes. That limit is 7.15 times the solar mass; beyond this limit the black holes cannot become neutron stars as those will be exploded in supernovae, before becoming stable neutron stars. Calculation sheets prepared based on the Gravitational Pressures and the Neutron Degeneracy Pressures of the cold stars showing the formation of neutron stars below the TOV limit (2.928 MΘ), concurrent formation of a neutron star and a black hole at the TOV limit, formation of black holes first and then becoming neutron stars from the cold stars having masses beyond the TOV limit, and black holes more massive than **7.15 MΘ** getting exploded have been presented in Tables 1, 2 & 3 respectively in Annexure-I below.

Annexure-I

Table-1

f	Radius (m)	Gr. Pr. (kg/m^2)	N. D.P (kg/m^2)	EV (m/s)	Gr. Energy (J)	Bind Energy J)	Remarks
1.5	10794	2.087E+33	2.087E+33	1.92E+08	3.3E+46	2.65E+47	Neutron Star
2.0	9805.8	5.449E+33	5.449E+33	2.33E+08	6.458E+46	3.53E+47	Neutron Star
2.5	9100	1.148E+34	1.148E+34	2.7E+08	1.087E+47	4.41E+47	Neutron Star
2.92	8646.2	1.921E+34	1.921E+34	2.99E+08	1.561E+47	51.5E+47	Neutron Star

Table-1 shows how the neutron stars are formed after the supernova explosion of red super giants and subsequent gravitational collapse of their cores for different solar masses. For a star of mass 1.5 MΘ the radius is a little above 10 kilometers only. At that radius (10794 m) its gravitational pressure counterbalances its neutron degeneracy pressure and at that equilibrium condition the star becomes a neutron star. Its escape velocity is less than the velocity of light and value of its gravitational energy is much less than its nuclear binding energy. These conditions imply that the newly formed neutron star is in stable state but is not a black hole.

Similar results have been obtained for 2 MΘ and 2.5 MΘ ; it is clear from the table that higher the mass, lesser is the radius of the neutron star. EVs of these neutron stars are less than the velocity of light. However, the EVs are tending towards the velocity of light. A neutron star of mass 2.92 MΘ, its EV is almost equal to the velocity of light.

If the mass of a cold star exceeds 2.92 times of solar mass, **approximately** at **2.928** MΘ as per the present calculations, following data emerge.

Table 2

f	Radius (m)	Gr. Pr. (kg/m^2)	N. D.P (kg/m^2)	EV (m/s)	Gr. Energy (J)	Bind Energy (J)	Remarks
2.9287	8634.2176	1.944E+34	1.944E+34	3E+08	1.573E+47	5.168E+47	NS & BH (TOV limit)

As it is clear from the above table that a cold star having mass of 2.928 times the solar mass (i.e. 5.825E+30 kg) is the least value at which a black hole (BH) is formed as the escape velocity of the star has equaled to the velocity of light. This black hole is also a neutron star as its gravitational pressure and the NDP are equal. But since at that condition a black hole has already been formed, the neutron star will remain invisible and cannot be detected even by powerful telescopes. This particular condition of star mass when the gravitational pressure of a cold star equals to its neutron degeneracy pressure and concurrently its escape velocity attains the speed of light is **the** Tolman, Oppenheimer and Volkoff (TOV) limit. It is around 2.928 MΘ as per the present calculations.

The radius at which the escape velocity of a cold star attains the velocity of light is its ‘Schwarzschild Radius’. At TOV limit value of Schwarzschild radius of the cold star is 8634.2176 metres (Approx.).

If the mass of the cold star exceeds the TOV limit, the cold star will become a black hole first. But since at black hole condition the NDP of the cold star is lower than its gravitational pressure, the black hole will continue to implode further till its gravitational pressure and NDP are balanced and ultimately at a reduced radius when the balance is reached the black hole will become a neutron star too. Table 3 figures show formation of black holes and afterwards becoming neutron stars from the cold stars having masses beyond TOV limit and up to 10 times solar mass.

Table-3

f	Radius (m)	Gr. Pr. (kg/m^2)	N. D.P (kg/m^2)	EV (m/s)	Gr. Energy (J)	Binding Energy (J)	Remarks
2.93	8638.0502	1.942E+34	1.941E+34	3E+08	1.573E+47	5.171E+47	Black Hole
2.93	8634.7	1.945E+34	1.945E+34	300058193	1.574E+47	5.171E+47	N S also B H
2.94	8667.5316	1.929E+34	1.919E+34	3E+08	1.579E+47	5.188E+47	Black Hole
2.94	8626	1.966E+34	1.966E+34	300721338	1.586E+47	5.188E+47	N S also B H
3.0	8844.42	1.852E+34	1.794E+34	3E+08	1.611E+47	5.295E+47	Black Hole
3.0	8568	2.103E+34	2.103E+34	304800872	1.663E+47	5.295E+47	N S also B H
4.0	11792.56	1.042E+34	6.876E+33	3E+08	2.148E+47	7.059E+47	Black Hole
4.0	7782.6	5.493E+34	5.493E+34	369286088	3.255E+47	7.059E+47	N S also B H
5.0	14740.7	6.669E+33	3.269E+33	3E+08	2.685E+47	8.824E+47	Black Hole
5.0	7225.4	1.155E+35	1.155E+35	428498554	5.478E+47	8.824E+47	N S also B H
6.0	17688.84	4.631E+33	1.78E+33	3E+08	3.222E+47	1.059E+48	Black Hole
6.0	6799	2.122E+35	2.122E+35	483891965	8.383E+47	1.059E+48	N S also B H
7.0	20636.98	3.402E+33	1.065E+33	3E+08	3.759E+47	1.235E+48	Black Hole
7.0	6457.99	3.548E+35	3.548E+35	536284991	1.201E+48	1.235E+48	N S also B H
7.14	21049.7196	3.27E+33	9.967E+32	3E+08	3.834E+47	1.260E+48	Black Hole
7.14	6417	3.786E+33	3.786E+33	543348400	1.258E+48	1.260E+48	N S also B H
7.15	21079.201	3.261E+33	9.921E+32	3E+08	3.84E+47	1.2618E+48	Black Hole
7.15	6414.2	3.804E+35	3.803E+35	543847427	1.2619E+48	1.2618E+48	SN Explosion
7.15	6411.6	3.81E+35	3.81E+35	543957685	1.262E+48	1.2618E+48	Already exploded
7.16	21108.6824	3.252E+33	9.874E+32	3E+08	3.845E+47	1.2636E+48	Black Hole
7.16	6423	3.793E+35	3.786E+35	543854663	1.264E+48	1.2636E+48	SN Explosion
7.16	6410.1	3.824E+35	3.824E+35	544401628	1.266E+48	1.2636E+48	Already exploded
7.17	21138.1638	3.243E+33	9.829E+32	3E+08	3.851E+47	1.2653E+48	Black Hole
7.17	6417	3.818E+35	3.812E+35	544488692	1.268E+48	1.2653E+48	SN Explosion
7.17	6407.4	3.841E+35	3.841E+35	544896434	1.27E+48	1.2653E+48	Already Exploded
7.18	21167.6452	3.234E+33	9.783E+32	3E+08	3.856E+47	1.267E+48	Black Hole
7.18	6425	3.81E+35	9.797E+35	544528935	1.27E+48	1.267E+48	SN Explosion
8.0	23585.12	2.605E+33	6.822E+32	3E+08	4.296E+47	1.4118E+48	Black Hole
8.0	7176.5	3.039E+35	2.615E+35	543856039	1.412E+48	1.4118E+48	SN Explosion
9.0	26533.26	2.058E+33	4.607E+32	3E+08	4.833E+47	1.5884E+48	Black Hole
9.0	8073	2.402E+35	1.767E+35	543874986	1.589E+48	1.5884E+48	SN Explosion
10.0	29481.4	1.667E+33	3.243E+32	3E+08	5.37E+47	1.7648E+48	Black Hole
10.0	8970.6	1.945E+35	1.243E+35	543856797	1.765E+48	1.7648E+48	SN Explosion

From the Table 3 following can also be derived:

1) Up to 7.14 times the solar mass (i.e. at 1.420E+31 kg) a collapsing cold star will become a black hole first and then it will become a neutron star at reduced radius (6417 metres) when its gravitational pressure will become equal to its neutron degeneracy pressure. All these cold stars (having masses up to 7.14 times the solar mass) will remain stable as black holes and as neutron stars simultaneously. However, these neutron stars will not be visible or detectable even by powerful telescopes as those are black holes too. It can also be inferred that maximum mass of a stable stellar black hole is 1.420E+31 kg (i.e. 7.14 MΘ). More massive stellar black holes are not stable and ultimately disintegrate after supernova explosions.
2) A cold star of mass 7.15 times the solar mass (1.422E+31 kg) will become a black hole but ultimately explode in a supernova before becoming a neutron star. At a radius of 6414.2 metres, the cold star, which has already become a black hole, will continue to collapse as its gravitational pressure is higher than the neutron degeneracy pressure, and eventually it will explode as its gravitational energy will be equal or exceed to its nuclear binding energy.
3) This pattern of phenomenon will be observed beyond the mass of 7.15 times the solar mass and in Table-3 calculations of cold stars up to 10 solar mass have been shown.

For cold rotating stars, calculated values shown above may marginally vary; however, the changes will be minimal and will not affect the findings much. But, if the density of the cold star varies while collapsing, a cold star of mass around 6.3 times the times the solar mass will explode in a supernova before becoming a neutron star.

DECLARATION STATEMENT

Funding	No, I did not receive.
Conflicts of Interest	No conflicts of interest to the best of our knowledge.
Ethical Approval and Consent to Participate	No, the article does not require ethical approval and consent to participate with evidence.
Availability of Data and Material	Not relevant
Authors Contributions	I am only the sole author of the article.

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Prosad Bhattacharya, B. Mech. Engg. from Jadavpur University, Kolkata. 34 years' experience in petroleum oil industry in Indian Oil Corporation Limited, a 'Fortune-500' and a 'Maharatna' company of India. Have varied experience in operation, maintenance, inspection, technical services and construction & commissioning of crude oil, petroleum products and gas pipeline transportation system. On official assignments, visited France, England, Sudan and Houston (Texas). After retirement from service in December 2014 as General Manager (Maintenance & Inspection), I devoted time in studying different branches of Physics. It was the pioneering work of revered India-born astrophysicist Nobel Laureate Subrahmanyan Chandrasekhar inspired me to study astrophysics. In order to understand astrophysics, I had to study nuclear physics and quantum mechanics. My study was mainly through the free online sources. An article titled 'Deep Space Conundrum Excites Scientists About Neutron Star That Might Also Be a Black Hole' published in the New York Times in 2020 motivated me to delve into this subject. My technical background and experience helped me in my research work which has been entirely a solo effort.
